NTNU

Kunnskap for en bedre verden

TMA4212 - Numerical solution of differential equations by difference methods

Project 2

Finite Element Methods

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1 Abstract

The Finite Element Method (FEM) is a powerful and widely used tool for solving partial differential equations (PDEs), particularly in industry-level applications. In this report, we investigate the convergence order of the FEM with respect to the H^1-norm for solving the Poisson equation. Numerical experiments are conducted to confirm these theoretical results. Furthermore, we extend the method's applicability to optimal control problems (OCPs), focusing on the efficient heating and cooling of a rod. This problem requires balancing the cost of heating with the accuracy of the resulting temperature distribution.

2 Introduction

In order to completely understand the real world and the physical phenomena that come with it, one needs to use mathematics. Fluid behavior, sound propagation, and thermal transport are just a few of the many phenomena that are commonly described using PDEs (Harish 2025). To solve these, numerical techniques need to be employed, and in this article we're going to take a closer look at one group of them, more specifically, the finite element methods (FEM).

This report is going to take a closer look at two one-dimensional problems: the Poisson equation and an optimal control problem. We're going to implement a numerical method to solve these problems, and then verify whether the analytical and numerical results coincide with the theoretical error for the method.

For the optimal control problem, we aim to see how our implementation solves a more real-life problem. Imagine one has a physical object, a rod for instance, and one aims to heat certain parts of the rod to a desired temperature. The challenge lies in determining the most efficient and cost-effective way to achieve this. By formulating this problem mathematically, one can express it as a system of PDE's and then use the FEM to systematically discretize and solve these equations, allowing us to find the optimal heating strategy.

3 Theory

To implement a solution to a PDE, the Galerkin method can be used. That is

$$\text{Find } u_h \in V_h \text{ such that } a(u_h, v) = F(v), \quad \forall v \in V_h$$

Where V_h is a finite-dimensional subspace of V. For the problems described in this report, V_h is defined by,

$$V_h = X_h^2 \cap H_0^1(\Omega = (0,1)).$$

Using, X_h^2, the second degree Lagrange finite element space:

$$X_h^2 = \{v \in \mathcal{C}^0(\Omega) : v|_K \in \mathbb{P}_2, \quad \forall K \in \mathcal{T}_h\},$$

where $\mathcal{T}_h$ is a partition of the domain. $H_0^1(\Omega)$ is the subspace of the Sobolev space $H^1(\Omega)$ for which the functions u uphold the boundary conditions $u(0) = u(1) = 0$.

3.1 Error Analysis

Using FEM to solve the Poisson equation will only provide us with a numerical result, not the analytical. Therefore, it is interesting to find the error, $||u - u_h||_{H^1}$, or more specifically, find an upper bound for the error.

To find the error between the analytical and numerical results, we will use Lemma 4.3 and Lemma 4.4 in (Curry 2018, p. 18-19)

In order to use Lemma 4.3, one first needs to prove that the bilinear function, $a(u,v)$, fulfills the Lax-Milgram theorem, which states that a needs to be continuous and coercive, (Curry 2018, p. 15). a is defined as

$$a(u,v) = \int_0^1 u_x v_x dx.$$

Now, we have

$$a(u,v) = \int_0^1 u_x v_x dx = \langle u_x, v_x \rangle_{L^2} \overset{C.S.}{\leq} ||u_x||_{L^2} \, ||v_x||_{L^2} \leq ||u||_{H^1} \, ||v||_{H^1},$$

which fulfills the continuity requirement of the Lax-Milgram theorem, with $M = 1$.

For coercivity, we get

$$a(v,v) = \int_0^1 v_x v_x dx = \langle v_x, v_x \rangle_{L^2} = |v|_{H^1}^2 \overset{Poincare}{\geq} C||v||_{H^1}^2$$

which proves that $a(\cdot,\cdot)$ is coercive.

Now, as we know that our $a(\cdot,\cdot)$ fulfills the LM-theorem, we can use Lemma 4.3, which gives us

$$||u - u_h||_{H^1} \leq \frac{M}{\alpha} ||u - v_h||_{H^1} \quad \forall \, v_h \in X_h^2 \, \cap H^1(\Omega)$$

Using Poincaré, we get

$$||u - u_h||_{H^1} \leq \frac{M}{\alpha} \frac{1}{C} |u - v_h|_{H^1}.$$

Now, assuming $f \in H^1(\Omega)$, we have that

$$-u''' = f' \in L^2(\Omega),$$

which means that

$$u \in H^3(\Omega).$$

Using the interpolant, $v_h = u_h^{(2)}$, where $u_h^{(2)}$ is the 2^{nd} order polynomial that interpolates the nodes u, we can use Lemma 4.4, and we obtain

$$\begin{aligned} ||u - u_h||_{H^1} &\leq \frac{M}{\alpha} \frac{1}{C} |u - v_h|_{H^1} = \frac{M}{\alpha} \frac{1}{C} |u - u_h^{(2)}|_{H^1} \\ ||u - u_h||_{H^1} &\leq \frac{M}{\alpha} \frac{1}{C} C_{1,2} h^2 |u|_{H^3}. \end{aligned}$$

Finally, combining the constants gives

$$||u - u_h||_{H^1} \leq K h^2 |u|_{H^3} = \mathcal{O}(h^2),$$

and one concludes that the error is of 2^{nd} order.

3.2 Optimal Control Problem

Imagine a rod of one unit length - defined by $\Omega = (0,1)$ - that is to be heated or cooled to obtain a desired temperature profile $y_d \in L^2(\Omega)$. It can be assumed that the (relative) temperature at the boundaries is zero. The heating is controlled by a source $u = u(x)$, and a cost $\alpha \in (0,\infty)$ associated with the heating/cooling. The goal is to minimize the discrepancy from y_d, and at the same time minimize the total cost of heating/cooling. The problem can be stated as an optimization problem.

$$\min_{u,y} J(y,u) = \min_{u,y} \frac{1}{2} \int_\Omega (y - y_d)^2 dx + \frac{\alpha}{2} \int_\Omega u^2 dx \tag{1}$$

$$\text{subject to } -\Delta y = u \text{ and } y(0) = y(1) = 0 \text{ (in the weak sense)}$$

This problem is an example of a PDE optimal control problem (OCP) with the control variable u, and the state variable y (desired outcome). Problem (1) can be approximated by the following finite element minimization problem:

$$\min_{u_h, y_h \in V_h} \frac{1}{2}||y_h - \overline{y}_d||_{L^2(\Omega)} + \frac{\alpha}{2}||u_h||^2_{L^2(\Omega)} \quad \text{s. t. } a(y_h, v) = \langle u_h, v\rangle_{L^2(\Omega)} \; \forall v \in V_h \tag{2}$$

where $\overline{y}_d$ is the interpolation of y_d onto X_h^2, $V_h = X_h^2 \cap H_0^1(\Omega)$, and a is the bilinear form given by

$$a(u, v) = \int_0^1 u_x v_x \, dx$$

We can interpret this as finding the unknown coefficients $u = (u_1, ..., u_{2N-1})$, $y = (y_1, ..., y_{2N-1})$ that solve the real-valued minimization problem

$$\min_{u, y \in \mathbb{R}^{2N-1}} G(y, u) \text{ subject to } By = Fu. \tag{3}$$

Letting $u, v, y \in V_h$, one can write

$$u = \sum_i u_i \phi_i(x), \quad y = \sum_i y_i \phi_i(x), \quad v = \sum_i v_i \phi_i(x)$$

for some finite basis $\phi_1, \phi_2, ..., \phi_m$.

Let now B, F be defined as

$$B_{ij} = \int_\Omega \phi_i'(x)\phi_j'(x)dx, \quad F_{ij} = \int_\Omega \phi_i(x)\phi_j(x)dx \tag{4}$$

Note that these are both symmetric matrices. It can then be shown that the inner product of u and v can be written as

$$\langle u, v\rangle_{L^2(\Omega)} = u^T F v = v^T F u \tag{5}$$

Furthermore, we can then write G as

$$G(y, u) = \frac{1}{2}(y^T F y + \overline{y}_d^T F \overline{y}_d - 2\overline{y}_d^T F y) + \frac{\alpha}{2} u^T F u$$

which then let's us write the Lagrangian as

$$\mathcal{L}(u, y, \lambda) = G(y, u) - \lambda^T B y + \lambda F u$$

By differentiation of the Lagrangian we can then state the optimality conditions as

$$\begin{aligned} \nabla_\lambda \mathcal{L} &= Fu - By = 0 \\ \nabla_y \mathcal{L} &= Fy - F\overline{y}_d - B\lambda = 0 \\ \nabla_u \mathcal{L} &= \alpha F u + F\lambda = 0 \end{aligned}$$

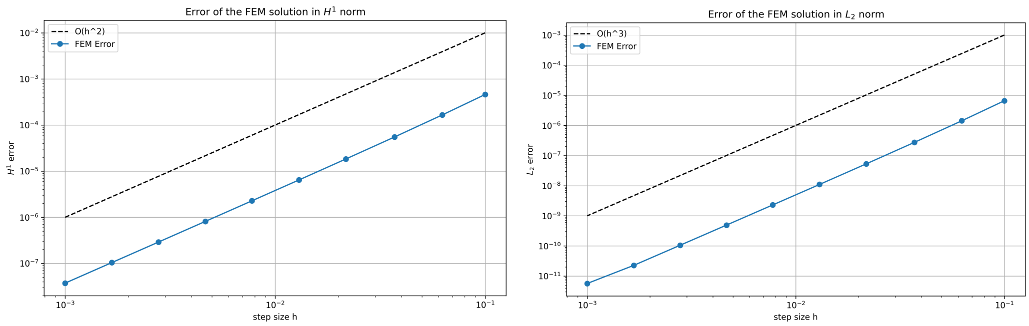
This linear system of equations can be set in the *standard* format $Ax = b$ by defining

$$A = \begin{bmatrix} F & -B & 0 \\ 0 & F & -B \\ \alpha F & 0 & F \end{bmatrix} \quad x = \begin{bmatrix} u \\ y \\ \lambda \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ F\overline{y}_d \\ 0 \end{bmatrix} \tag{6}$$

4 Numerical Experiments

4.1 Error verification

Now that we have found the theoretical error between the solution, u, and the numerical approximation, u_h, let's verify this numerically. By comparing our theoretical error on the H^1 and L^2 norms to a 2nd and 3rd order function, respectively, we managed to verify our numerical results for H^1. See figure 1.



(a) Error with respect to the H1-norm

(b) Error with respect to the L2-norm

Figure 1: Showing the numerical error between exact solution and numerical solution, $||u - u_h||$, on the norms L^2 and H^1.

4.2 Example: Optimal Control Problem

The optimal control problem described in problem (1) was solved for three different ideal temperature profiles.

$$y_{d_1} = \frac{1}{2}x(1-x), \quad y_{d_2} = 1, \quad y_{d_3} = \begin{cases} 1 & \text{for } x \in [\frac{1}{4}, \frac{3}{4}] \\ 0 & \text{otherwise} \end{cases}$$

Notice that only y_{d_1} is in the Sobolev-subspace $H_0^1(\Omega)$, as y_{d_2} does not follow the boundary conditions, and y_{d_3} does not have a square integrable weak derivative.

To solve the problems we used cost parameters $\alpha \in \{10^{-1},\ 10^{-4},\ 10^{-7}\}$. The results can be seen in figure 2. It is apparent that the lower the cost parameter, the more accurate the result is. This holds for all three cases of y_d. Further verification of this can be seen in figure 4 which clearly shows how the temperature discrepancy scales with an increased α.

For the lowest α-value, the calculated temperature profile is very similar to the ideal one. However, for the two other values for the cost parameter, one can see that the temperature profile is not as accurate. That being said, it seems that the plots "make sense", with an exception for y_{d_2}: here one was supposed to have a symmetric temperature profile, so naturally one would assume that the heat would be distributed symmetrically, and that the end result in the temperature would be symmetric too. This is not the case! It can be seen that the simulations suggest that one should heat the left part of the rod, while simultaneously cool the right part. This is most evident for the lowest cost-parameter, although strangely this gives the most accurate result.

In figure 3 one can see how the total cost decreases with an increasing value for α. Again, an exception is y_{d_2}, where one does not have a strict decrease.

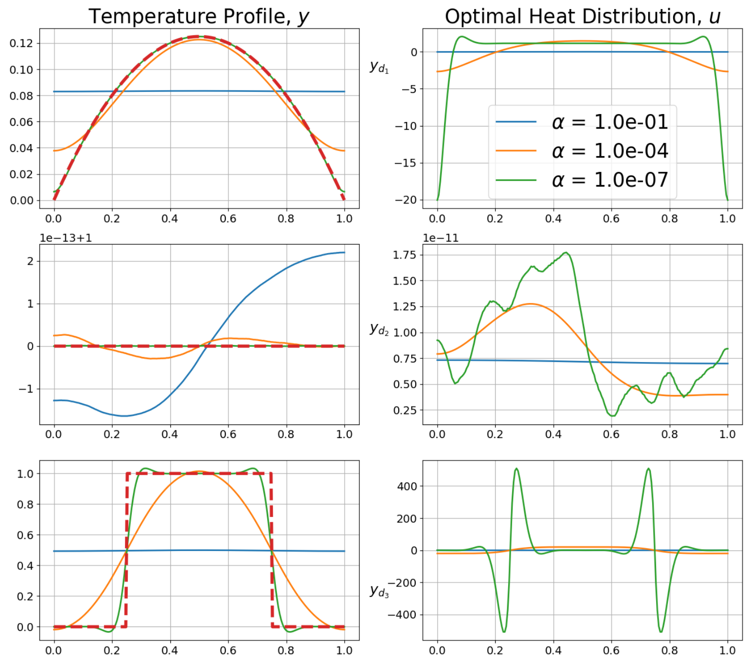


Figure 2: Visualization of the optimal heat distribution and the resulting temperature profile for three different ideals (red stipled line)

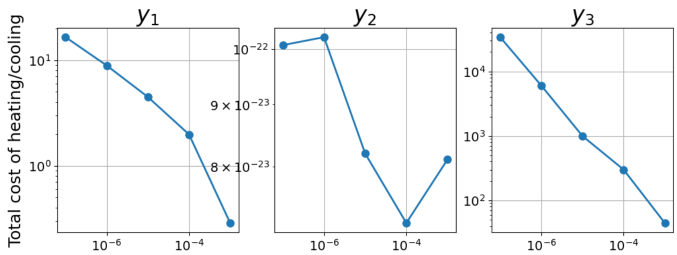


Figure 3: Showing how different values of α affect the total cost of heating/cooling

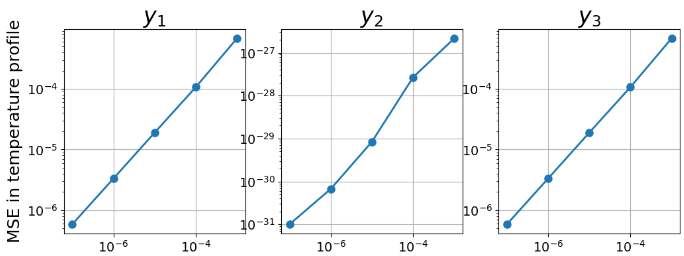


Figure 4: Showing how different values of α affect the discrepancy from the ideal temperature profile

5 Conclusion

In this report, we have explored the use of Galerkin methods for solving the Poisson equation and an optimal control problem.

Particularly, we analyzed a finite element method applied to the Poisson problem, and showed that the error is of 2^{nd} order. The same method was then applied to solve an optimal control problem. The solution of the optimization problem showed how one should effectively heat up a rod of length one, to achieve a preset temperature profile. Furthermore, it was shown that this varied based on the cost of heating up.

Future extensions of this work could involve solving the same optimal control problem, but extended to two or three dimensions. Furthermore, one could set restrictions such as where the rod can be heated. Another potential improvement would be to set up and solve the system of equations for the case where the rod is heterogeneous.

Bibliography

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Harish, A. (2025). *Finite Element Mehod - What is it? FEM and FEA Explained.* URL: https://www.simscale.com/blog/what-is-finite-element-method/ (visited on 2nd Apr. 2025).

Appendix

A AI Declaration

In conducting this project we've taken use of general artificial intelligence (GAI) through the use of ChatGPT. We have not used software such as "Github Copilot" for generating code.

In summary, the usage of ChatGPT has been restricted to:

- Proofreading of written text and suggestions of alternative formulations.
- Suggestion of bug-fixes for erroneous code.
- Finding sources, much like one would use a search engine like *Google*.

This means that we *have not* used GAI for:

- Generation of new text for this report.
- Generation of new code.
- Mathematical analysis of the PDE and it's properties.
- Mathematical analysis of the numerical methods used.